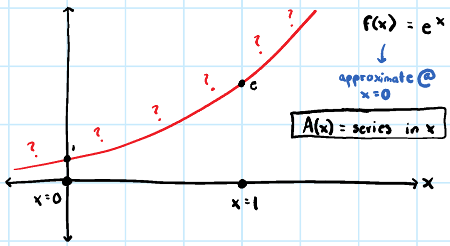
Taylor Series

Wednesday, April 19, 2023 8:56 AM



$f(x) = e^x$ ($\leftarrow$ how to evaluate)

only know how to evaluate polynomials

ideal algorithm:

1) given function $f(x)$ & a value $x=a$, check if $f(x)$ is differentiable near a

different slope approaching 0

$|x|$ @ $x=0$

$\sqrt{x}$ @ $x=0$

(not differentiable)

not defined @ interval that contains 0

2) apply taylor's thm to approximate $f(x)$ thru power series $\sum_{n=0}^{\infty} a_n (x-a)^n$ centered @ $\boxed{x=a}$

3) evaluate series where you can (compute radius of convergence)

taylor's thm: if $f(x)$ differentiable @ $x=a$, then $f(x) = \sum_{n=0}^{\infty} \underbrace{\frac{f^{(n)}(a)}{n!}}_{\#\, a_n} \cdot (x-a)^n$ near $x=a$

where it converges

* never expand series @ point a *

expanded $f(x) = \underbrace{f(a)}_{n=0} + \underbrace{f'(a)(x-a)}_{n=1} + \underbrace{\frac{f''(a)}{2}(x-a)^2}_{n=2} + \dots$

linear in taylor (tangent line) — quadratic

ex 1) evaluate e, i.e. $f(1)$ if $f(x) = e^x$

taylor expansion of $f(x) = e^x$ @ $x=0$ is

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$f^{0}(x) = e^x = e^0 = 1$ (a_0)

$f'(x) = e^x = e^0 = 1$ (a_1)

$f''(x) = e^x = e^0 = 1$ (a_2)

$f'''(x) = e^x = e^0 = 1$ (a_3)

$f^{(4)}(x) = e^x = e^0 = 1$ (a_4)

check right hand side has $R = \infty$ radius of convergence (ratio test)

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{x^{n+1}}}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{x^n}} \right| = \lim_{n\to\infty} \left| \frac{x}{(n+1)} \right| = 0 \quad \boxed{R = \infty}$$

always approaching 0 no matter x

then $f(1) = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$

1, 2, 2.5, $2.5 + \frac{1}{6} = 2.66$, 2.7

$e \approx 2.71828$

ex 2) $f(x) = \frac{1}{1-x}$, we know $f(\frac{1}{2}) = \frac{1}{1-\frac{1}{2}} = 2$

$f(5) = \frac{1}{1-5} = -\frac{1}{4}$

taylor series @ $x=0$ is ...

$\frac{1-x^{n+1}}{1-x}$

$f^{(0)}(x) = \frac{1}{1-x} = 1$ ($x=0$)

$f'(x) = \frac{1}{(1-x)^2} = 1$

taylor series @ $x=0$ is ...

$$f(x) = \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{n!}{n!}x^n = \sum_{n=0}^{\infty} x^n = \underbrace{1 + x + x^2 + x^3 + x^4 + \ldots}_{\text{geometric series}}$$

$\frac{1-x^{n+1}}{1-x}$

$f^{(n)}(0) = n!$

$f^{(0)}(x) = \frac{1}{1-x} = 1$

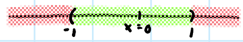
$f'(x) = \frac{1}{(1-x)^2} = 1$

$f''(x) = 2\,\frac{1}{(1-x)^3} = 2$

$f'''(x) = 2 \cdot 3 \cdot \frac{1}{(1-x)^4} = 3!$

$f^{(n)}(x) = n! \cdot \frac{1}{(1-x)^{n+1}} = n!$

radius of convergence is $R=1$



$\sum_{n=0}^{\infty} x^n \rightarrow r = |x| < 1 \rightarrow$ converge

(geometric series)

thus $A(x) = \sum_{n=0}^{\infty} x^n$ only makes sense for $|x| < 1$

1) @ $x = 1/2$ $\quad f(1/2) = 2 \quad \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = 2$ ✓ (inside interval of convergence)

2) @ $x = 5$ $\quad f(5) = -1/4$, but RHS = taylor's series @ $x=5 \rightarrow \infty$ ✗ (outside interval)

too big of r value

$1 + 5 + 5^2 + 5^3 + 5^4 \ldots$